Journal of Differential Equations **176**, 290–314 (2001)
doi:10.1006/jdeq.2000.3964, available online at http://www.idealibrary.com on IDEAL®

Compressible Stokes Problem on Nonconvex Polygonal Domains

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Received January 24, 2000; revised June 12, 2000

The compressible Stokes system with the inflow boundary condition is studied on the polygonal domain D. It is shown that the lowest order corner singularity of the system is the same as that of the Laplacian. The velocity $\mathbf{u}$ is split into a singular and regular part near each concave vertex. If the polygon is convex, it is shown that $\mathbf{u} \in (H^2(D))^2$. © 2001 Academic Press

Key Words: compressible flows; nonconvex polygon; singular function.

1. INTRODUCTION AND MAIN RESULTS

A complete mathematical understanding of boundary value problems for the compressible Navier–Stokes equations or the compressible Stokes equations has not yet been achieved (see [2, 7, 8]). A problem not yet considered is to analyse the equations on a plane polygonal domain. The purpose of this paper is to study a very simple form of the compressible Stokes system on a (convex or nonconvex) polygonal domain and in particular to provide a decomposition of the velocity field into singular and regular parts with the singular part determined by certain harmonic

[1] This author is partially supported by Com^{2}MaC-KOSEF.

0022-0396/01 $35.00
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functions near the concave vertices. The equations to be considered (see [7, 8]) are

$$\begin{cases} -\mu\, \Delta \mathbf{u} + \nabla p = \mathbf{f} & \text{in } D, \\ \operatorname{div} \mathbf{u} + \mathbf{U} \cdot \nabla p = g & \text{in } D, \\ \mathbf{u} = 0 & \text{on } \partial D, \\ p = 0 & \text{on } \partial D_{in}, \end{cases} \tag{1.1}$$

where D is an open bounded domain in the plane with polygonal boundary ∂D, $\mathbf{u} = [u, v]$ is the unknown velocity vector, and p is the unknown pressure; $\mathbf{f}$ and g are given functions, $\mathbf{U} = [1, 0]$ and $\mu > 0$ is the coefficient of bulk viscosity. The inflow and outflow boundaries, ∂D_{in} and ∂D_{out}, are defined by

$$\begin{aligned} \partial D_{in} &= \{(x, y) \in \partial D : \mathbf{U} \cdot n < 0\}, \\ \partial D_{out} &= \{(x, y) \in \partial D : \mathbf{U} \cdot n \geqslant 0\}, \end{aligned} \tag{1.2}$$

where $n = [n_1, n_2]$ denotes the unit outward pointing normal to ∂D.

The equations in (1.1) may be obtained by linearizing the barotropic Navier–Stokes equations around the ambient flow $\mathbf{U}$ and dropping the convective term in the linearized momentum equation. For details see [7]. For simplicity, it is assumed that the shear viscosity vanishes. The pressure boundary condition in (1.1d) arises from the hyperbolic nature of the continuity equation (1.1b) and the fact that the ambient flow field $\mathbf{U} \neq \mathbf{0}$ on ∂D.

The theory of corner singularities for the Poisson equation is contained, for example, in [3], and regularity results for the (incompressible) Stokes system are given in [6]. A similar analysis has not been given for the compressible Stokes system, which is the purpose of this paper. Hopefully such results will be of use in studying the numerical solution of the compressible Navier–Stokes equations in regions with corners.

For an existence and regularity result for problem (1.1) on a smooth domain the following theorem may be found in [7, Theorem 2.1].

THEOREM 1.1. *Let $2 < q < 3$ and suppose ∂D is smooth. Assume that $\mathbf{f} \in L^q(D)$ and $g \in H^{1,q}(D)$. Then there is a unique solution $[\mathbf{u}, p] \in (H_0^1(D))^2 \times L^2(D)$ of problem* (1.1). *Furthermore there is a constant $C = C(D, \mu)$ such that if μ is large enough,*

$$\|\mathbf{u}\|_{H^{2,q}(D)} + \|p\|_{H^{1,q}(D)} \leqslant C(\|\mathbf{f}\|_{L^q(D)} + \|g\|_{H^{1,q}(D)}). \tag{1.3}$$

We now state the main results of this paper. The proofs are given in Sections 2 and 4.

THEOREM 1.2. *Let D be a nonconvex polygonal domain. Suppose μ is large enough. Let $\mathbf{f} \in (L^2(D))^2$, $g \in H^1(D)$ and let $[\mathbf{u}, p]$ be a weak solution of* (1.1) *in the sense of* (2.4). *The velocity* $\mathbf{u}$ *can be split into a singular and regular part, $\mathbf{u} = \mathbf{u}_s + \mathbf{u}_r$, with the property that $\mathbf{u}_r \in (H^2(D))^2 \times H^1(D))$ and near each concave vertex, with interior angle $\omega > \pi$, $\mathbf{u}_s = \mathbf{C}\, r^\alpha \sin[\alpha(\theta - \omega_1)]$. Here r is the distance to the vertex, $\alpha = \frac{\pi}{\omega} < 1$, the angle ω_1 is defined in section 2, and $\mathbf{C}$ is a constant vector which is given in* (2.52). *Furthermore there is a constant $C = C(D, \mu)$ such that if μ is large enough,*

$$\|\mathbf{u}_r\|_{2,D} + \|p\|_{1,D} \leqslant C(\|\mathbf{f}\|_{0,D} + \|g\|_{1,D}). \tag{1.4}$$

From Theorem 1.2 we see that the singular exponent α for the compressible Stokes system (1.1) is completely determined by the singular exponent of the velocity in the momentum equation (1.1a), i.e., is the same as the singular exponent of the Laplacian problem. In contrast with the Stokes system, there is no singular function for the pressure at this level in a singular expansion of the solution.

Next we state a regularity result for system (1.1) on a convex polygon, which is proven in Sections 3 and 4.

THEOREM 1.3. *Let D be a convex polygon. Suppose μ is large enough. Let $\mathbf{f} \in (L^2(D))^2$, $g \in H^1(D)$ and let $[\mathbf{u}, p]$ be a weak solution of* (1.1) *in the sense of* (2.4). *Then $[\mathbf{u}, p] \in (H^2(D))^2 \times H^1(D))$ and there is a constant $C = C(D, \mu)$ such that*

$$\|\mathbf{u}\|_{2,D} + \|p\|_{1,D} \leqslant C(\|\mathbf{f}\|_{0,D} + \|g\|_{1,D}). \tag{1.5}$$

The main strategy used in Theorem 1.2 is based on splitting the solution into singular and regular parts and applying to our problem (1.1) known results for the Poisson problem on a polygonal domain. To construct the singular part of the solution we first pick a vector function, each component of which is a multiple of the harmonic function ϕ (see (2.2)) near the corner points. This function belongs to H^1 but not H^2. Second, for the divergence of the vector function we solve for the solution of a first order partial differential equation (see (2.7)), which may be regarded the singular part of the pressure, corresponding to the singular part of the velocity function. Note that the equation takes a form like the continuity equation. Hence the singular part of pressure can be represented in terms of the divergence of the vector function which consists of the harmonic function (see (2.8)). Because we are considering only a low order expansion of the solution, this singular part of the pressure does not appear in the final decomposition.

In order to prove Theorems 1.2 and 1.3 it suffices to analyze the behavior of the solution near each vertex of D. To do this, we study the problem in an infinite sector Ω whose vertex is placed at P, and whose sides are the extension to infinity of the two sides of $\partial\Omega$ which meet at P. Without loss of generality we assume that the vertex P is placed at the origin (0, 0). Let $\chi \in C_0^\infty(R^2)$ be a smooth cutoff function which is identically 1 in a neighborhood of the origin (0, 0), and which satisfies

$$\chi(x, y) \equiv 0 \qquad \text{for} \quad r = \sqrt{x^2+y^2} \geqslant r_0. \tag{1.6}$$

Without loss of generality we may assume that $r_0 = 1$. To investigate the behavior of the solution near the origin (0, 0) we obtain from (1.1) the following generalized compressible Stokes system:

$$\begin{cases} -\mu\,\Delta(\chi\,\mathbf{u}) + \nabla(\chi p) = \chi\mathbf{f} - 2\,\mu\nabla\chi\cdot\nabla\mathbf{u} - \mu\mathbf{u}\,\Delta\chi + p\,\nabla\chi & \text{in } \Omega, \\ \operatorname{div}(\chi\mathbf{u}) + \mathbf{U}\cdot\nabla(\chi p) = \chi g + \mathbf{u}\cdot\nabla\chi + p\,\mathbf{U}\cdot\nabla\chi & \text{in } \Omega, \\ \chi\mathbf{u} = 0 \qquad \text{on } \Gamma \text{ and on } \quad r = 1, \\ \chi p = 0 \qquad \text{on } \Gamma_{in} \text{ and on } \quad r = 1, \end{cases} \tag{1.7}$$

where Γ is the boundary of Ω and Γ_{in} is the incoming portion of Γ. It easily follows that $\chi\,\mathbf{u} \in (H^2_{loc}(\Omega))^2$ and $\chi\,p \in H^1_{loc}(\Omega)$ and the first two equations of the right hand sides of (1.7) are in $(L^2(\Omega))^2$ and $H^1(\Omega)$, respectively, and $[\chi\,\mathbf{u}, \chi\,p]$ is a generalized solution of the generalized compressible Stokes equations (1.7). If this construction is made for each vertex P_i of D, $1 \leqslant i \leqslant N$, we see that the original solution $[\mathbf{u}, p]$ of (1.1) may be expressed in the form

$$\mathbf{u} = \sum_{i=0}^{N} \mathbf{u}_i, \quad p = \sum_{i=0}^{N} p_i.$$

Here the function $[\mathbf{u}_i, p_i]$, $1 \leqslant i \leqslant N$, is the generalized solution of the form (1.7) corresponding to the vertex P_i of D and the remaining couple $[\mathbf{u}_0, p_0]$ is a generalized solution of the generalized compressible Stokes problem which vanishes in a neighborhood of the vertices of D. Using Theorem 1.1 we have $[\mathbf{u}_0, p_0] \in (H_0^2(D))^2 \times H_0^1(D)$. For details see Section 4.

For the proof of Theorem 1.2 on a nonconvex bounded polygonal domain it suffices to apply Theorem 1.2 to each concave vertex and apply Theorem 1.3 to each convex vertex and use the above mentioned estimate for $[\mathbf{u}_0, p_0]$.

We have thus reduced the proof of Theorem 1.2 to proving the following:

THEOREM 1.4. *Let Ω be a concave sector whose vertex is placed at the origin. Suppose μ is large enough. Let $[\mathbf{u}, p]$ be a function in Ω which satisfies $\mathbf{u} \in (H_0^1(\Omega) \cap H_{loc}^2(\Omega))^2$, $p \in L^2(\Omega) \cap H_{loc}^1(\Omega)$, and $\mathbf{u} \equiv 0$, $p \equiv 0$ for $r > 1$, and in both the generalized and pointwise sense, $[\mathbf{u}, p]$ satisfies*

$$\begin{cases} -\mu\, \Delta \mathbf{u} + \nabla p = \mathbf{f} \in (L^2(\Omega))^2, \\ \operatorname{div} \mathbf{u} + \mathbf{U} \cdot \nabla p = g \in H^1(\Omega), \\ \mathbf{u}|_\Gamma = 0, \qquad p|_{\Gamma_{in}} = 0. \end{cases} \tag{1.8}$$

Then the velocity $\mathbf{u}$ can be split into the singular and regular parts near the origin of Ω: $\mathbf{u} = \mathbf{u}_s + \mathbf{u}_r$ with $\mathbf{u}_s$ given in Theorem 1.2, and $[\mathbf{u}_r, p] \in (H^2(\Omega))^2 \times H^1(\Omega)$. Furthermore there is a constant $C = C(\Omega, \mu)$ such that

$$\|\mathbf{u}_r\|_{2,\Omega} + \|p\|_{1,\Omega} \leqslant C(\|\mathbf{f}\|_{0,\Omega} + \|g\|_{1,\Omega}). \tag{1.9}$$

Likewise the proof of Theorem 1.3 is reduced to the following result:

THEOREM 1.5. *Let Ω be a convex sector whose vertex is placed at the origin. Suppose μ is large enough. Then there is a unique solution $[\mathbf{u}, p]$ of (1.8) and a constant $C = C(\Omega, \mu)$ such that*

$$\|\mathbf{u}\|_{2,\Omega} + \|p\|_{1,\Omega} \leqslant C(\|\mathbf{f}\|_{0,\Omega} + \|g\|_{1,\Omega}). \tag{1.10}$$

In Theorems 1.2–1.5 we have assumed that the parameter μ is sufficiently large. This assumption is not needed for the existence of a solution to the system (1.1); since the coefficients μ and $\mathbf{U}$ are constant, an integration by parts and use of the Lax–Milgram lemma easily gives existence and uniqueness. The requirement is needed in the proof of Lemma 2.3. Precisely how large μ should be depends on the various angles in Ω, and on the vector $\mathbf{U}$. In addition, the proof of Lemma 2.5 requires that μ not equal a certain finite set of numbers. (Two numbers must be excluded for each vertex of Ω.) It would be interesting to know if these restrictions on μ are necessary.

To apply to our problem (1.1) known results for the Poisson problem on polygonal domain we define solution operators as follows. We define A: L^2 (or H^{-1}) $\rightarrow H^1$ by $z := AF$ where z is the solution of

$$\begin{cases} -\Delta z = F & \text{in } \Omega, \\ z = 0 & \text{on } \Gamma. \end{cases} \tag{1.11}$$

We let $Q = \{q \in L^2(D) : \|q\|_{Q,D} < \infty\}$ be the space normed by

$$\|q\|_{Q,D} = \sqrt{\|q\|_{0,D}^2 + \|\mathbf{U} \cdot \nabla q\|_{0,D}^2}$$

and define the operator B: $L^2 \longrightarrow Q$ by $q := BG$ where q is the solution of

$$\begin{cases} q_x = G & \text{in } \Omega, \\ q = 0 & \text{on } \Gamma_{in}. \end{cases} \tag{1.12}$$

The set $\Gamma_{in} = \{(x, y) \in \Gamma : \mathrm{U} \cdot n < 0\}$ with $\mathrm{U} = [1, 0]$ and if the set $\{(\delta(y), y) : -\infty < y < \infty\}$ describes the incoming portion Γ_{in} of Γ, the solution q of (1.12) can be represented by

$$q(x, y) = \int_{\delta(y)}^{x} G(s, y)\, ds. \tag{1.13}$$

We define the trace operator γ of H^1-functions on the incoming portion Γ_{in} as

$$\gamma G := G(\delta(y), y) = G|_{\Gamma_{in}}. \tag{1.14}$$

From the operators defined in (1.12)–(1.14) we observe

$$\nabla_x q = \nabla_x BG = G, \tag{1.15}$$

$$\nabla_y q = \nabla_y BG = B\, \nabla_y G - \gamma G\, \delta'(y), \tag{1.16}$$

where $\nabla_x := \frac{\partial}{\partial x}$, $\nabla_y := \frac{\partial}{\partial y}$, and

$$\delta'(y) = \begin{cases} m_1 & (y > 0), \\ m_2 & (y < 0), \end{cases} \tag{1.17}$$

where m_i (i$=$1, 2) are the real numbers.

The paper is organized as follows: In Section 2, Theorem 1.4 is shown for the nonconvex sector and in Section 3, Theorem 1.5 is shown for the convex sector. In Section 4, combining the results demonstrated in Sections 2 and 3 suitably, Theorem 1.2 and 1.3 are shown.

In what follows, we use $L^2(D)$ the space of the square integrable functions on D with the norm denoted by $\|u\|_{0,D}$. The space $H^k(D)$ denotes the space of all functions which are square integrable up to the k-th order derivatives and with norm denoted by $\|u\|_{k,D}$. We set $H_0^k(D) = H^k(D) \cap H_0^1(D)$. We denote by $H^{-1}(D)$ the dual space of $H_0^1(D)$ normed by

$$\|f\|_{-1,D} = \sup_{0 \neq v \in H_0^1(D)} \frac{\langle f, v \rangle}{\|v\|_{1,D}},$$

where $\langle\, ,\rangle$ denote the duality pairing. If u belongs to $H^k(\mathcal{O})$ for every measurable compact subset $\mathcal{O}$ of D we say that u is locally in $H^k(D)$ and we write $u \in H^k_{loc}(D)$.

2. NONCONVEX SECTOR

In this section Theorem 1.4 is proven. We will analyze the problem (1.8) on the following concave sector Ω defined by

$$\Omega = \{(x, y) \mid -2\pi < \theta < \omega_1, 0 \leqslant \theta < \omega_2, 0 < r < \infty\}, \tag{2.1}$$

where $\theta = \tan^{-1}\frac{y}{x}$, $r = \sqrt{x^2+y^2}$ and $-\pi < \omega_1 < -\frac{\pi}{2}, \frac{\pi}{2} < \omega_2 < \pi$. For other cases of nonconvex sector (see (2.61)–(2.63)) results similar to those obtained in this section can be shown by using the same procedures.

We first cite the following result for the Poisson problem (1.11) on the domain Ω specified in (2.1), which will be useful in our analysis (see [5]):

THEOREM 2.1. *Let Ω be the sector given in* (2.1). *Assume that $z \in H^1_0(\Omega)$ is the solution of* (1.11) *in both the weak and pointwise sense with $z \equiv 0$ for $r = \sqrt{x^2+y^2} \geqslant 1$ and that $F \in L^2(\Omega)$. Then the solution z can be split into singular and regular parts: $z = \Lambda(F)\,\phi + w$, $w \in H^2(\Omega)$ with $\|w\|_{2,\Omega} \leqslant C\,\|F\|_{0,\Omega}$ and*

$$\phi(x, y) := \chi(r)\,\psi(x, y) = \chi(r)\,r^\alpha \sin[\alpha(\theta - \omega_1)], \qquad \alpha = \frac{\pi}{\omega} < 1, \tag{2.2}$$

$$\Lambda(F) = \frac{1}{\pi}\iint_\Omega r^{-\alpha}\sin[\alpha(\theta-\omega_1)]\,F(x, y)\,dx\,dy, \tag{2.3}$$

where $\omega_1 < \theta = \tan^{-1}\frac{y}{x} < \omega_2$ and ω_i $(i=1,\ 2)$ are defined in (2.1) *and $\omega = \omega_2 - \omega_1$.*

For a weak formulation for the problem (1.8) we define three bilinear forms a, b, and c:

$$a(\mathbf{u}, \mathbf{v}) = \mu\int_\Omega \nabla\mathbf{u}\cdot\nabla\mathbf{v}\,\mathbf{dx}, \qquad \mathbf{u}, \mathbf{v} \in (H^1_0(\Omega))^2,$$

$$b(\chi, \mathbf{v}) = -\int_\Omega \chi\,\operatorname{div}\mathbf{v}\,\mathbf{dx}, \qquad \chi \in L^2(\Omega), \mathbf{v} \in (H^1_0(\Omega))^2,$$

$$c(p, q) = \int_\Omega \mathbf{U}\cdot\nabla p\,q\,\mathbf{dx}, \qquad p \in Q, q \in L^2(\Omega).$$

Using these bilinear forms, the weak formulation for (1.8) is to find $[\mathbf{u}, p] \in (H_0^1(\Omega))^2 \times L^2(\Omega)$ such that

$$\begin{cases} a(\mathbf{u}, \mathbf{v}) + b(p, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle, & \forall \mathbf{v} \in (H_0^1(\Omega))^2, \\ c(p, q) - b(\mathbf{u}, q) = \langle g, q \rangle, & \forall q \in L^2(\Omega). \end{cases} \tag{2.4}$$

In the next Lemma we give a bound for a solution $[\mathbf{u}, p]$ that has compact support.

LEMMA 2.1. *There is a constant $C > 0$ such that if $[\mathbf{u}, p]$ vanishes for $r \geqslant 1$ and satisfies* (1.8) *in both the generalized and pointwise sense, then* $\mu \|\mathbf{u}\|_{1,\Omega} + \|p\|_{0,\Omega} \leqslant C(\|\mathbf{f}\|_{0,\Omega} + \|g\|_{0,\Omega})$.

Proof. Setting $\mathbf{v} = \mathbf{u}$ and $q = p$ in (2.4) and adding, we get

$$\mu \int_\Omega |\nabla \mathbf{u}|^2 \, d\mathbf{x} + \int_{\Gamma_{out}} p^2 \, \mathbf{U} \cdot n \, ds = \int_\Omega \mathbf{f} \cdot \mathbf{u} + gp \, dx. \tag{2.5}$$

Using (1.12) with $G = g - \operatorname{div} \mathbf{u}$ and the boundedness of the solution operator B, we get $\|p\|_{0,\Omega} \leqslant \|B\| \, (\|\operatorname{div} \mathbf{u}\|_{0,\Omega} + \|g\|_{0,\Omega})$. Applying this to the right hand side of (2.5) and using Schwarz's inequality and the Poincaré inequality the inequality follows. ∎

On the basis of Theorem 2.1 we are going to split the solution of (1.8) into singular and regular parts and investigate its behavior near the corner point (0, 0). Recall that we focus on the local behavior of the solution near the concave vertex and assume that $\mathbf{u} \equiv 0$ and $p \equiv 0$ for $r \geqslant 1$. Thus, from (1.8) we see that $\mathbf{f} \equiv 0$ and $g \equiv 0$ for $r \geqslant 1$. We define

$$\begin{cases} \mathbf{u} = \mathbf{u}_s + \mathbf{u}_r, & \mathbf{u}_s = \mathbf{C}\phi, \\ p = p_s + p_r \end{cases} \tag{2.6}$$

where ϕ is defined in (2.2) and the unknown constant $\mathbf{C} = [C_1, C_2]$ will be determined later and p_s will be constructed shortly. Assuming that $\mathbf{C}$ is given, we let p_s be the solution of the following problem:

$$\begin{cases} p_{s,x} = -\operatorname{div} \mathbf{u}_s & \text{in } \Omega, \\ p_s = 0 & \text{on } \Gamma_{in}. \end{cases} \tag{2.7}$$

Note that $p_s \equiv 0$ for $r = \sqrt{x^2 + y^2} \geqslant 1$. Using the operator B defined in (1.12) the solution p_s of (2.7) is given by

$$p_s = -B \operatorname{div} \mathbf{u}_s. \tag{2.8}$$

Note that $\operatorname{div} \mathbf{u}_s = \mathbf{C} \cdot \nabla\phi$ and $B \operatorname{div} \mathbf{u}_s = \mathbf{C} \cdot B\, \nabla\phi$. The function p_s is a new ingredient in our analysis and it may have a certain physical significance. For a better understanding of the function p_s we give pictures (see Figs. 1 and 2) for the functions $-\mathbf{C} \cdot \nabla\psi$ and $-\mathbf{C} \cdot B\, \nabla\psi$ on a specified nonconvex sector: $\Omega_* = \{(x, y) \mid -2\pi < \theta < -\frac{3\pi}{4}, 0 \leqslant \theta < \frac{3\pi}{4}, |x| < 1, |y| < 1\}$ and $\omega = \frac{3\pi}{2}$, $\alpha = \frac{2}{3}$.

In the next lemma we give derivative bounds for p_s.

LEMMA 2.2. *The solution p_s of* (2.7) *satisfies the inequality*

$$\|p_s\|_{1, \Omega} \leqslant K_1\, |\mathbf{C}|, \tag{2.9}$$

where K_1 depends on α and ω.

Proof. To establish (2.9) we first compute the derivatives of $\phi = \chi\, \psi$. The derivatives of $\psi(x, y) := r^\alpha \sin[\alpha(\theta - \omega_1)]$ are given by

$$\psi_x(x, y) = \alpha(x^2 + y^2)^{\frac{\alpha}{2} - 1} \{x \sin[\alpha(\theta - \omega_1)] - y \cos[\alpha(\theta - \omega_1)]\}, \tag{2.10}$$

$$\psi_y(x, y) = \alpha(x^2 + y^2)^{\frac{\alpha}{2} - 1} \{y \sin[\alpha(\theta - \omega_1)] + x \cos[\alpha(\theta - \omega_1)]\} \tag{2.11}$$

and

$$\begin{aligned} |\nabla\psi(x, y)| &\leqslant \sqrt{2}\, \alpha(x^2 + y^2)^{\frac{\alpha}{2} - 1} \{|x| + |y|\} \\ &\leqslant 2\alpha(x^2 + y^2)^{\frac{\alpha}{2} - 1} (x^2 + y^2)^{\frac{1}{2}} \\ &\leqslant 2\alpha(x^2 + y^2)^{\frac{\alpha - 1}{2}}, \qquad \forall (x, y) \in \Omega. \end{aligned} \tag{2.12}$$

Furthermore,

$$\begin{aligned} \psi_{xy}(x, y) = {}& \alpha(\alpha - 2)\, (x^2 + y^2)^{\frac{\alpha}{2} - 2} \{xy \sin[\alpha(\theta - \omega_1)] + x^2 \cos[\alpha(\theta - \omega_1)]\} \\ &+ \alpha^2 (x^2 + y^2)^{\frac{\alpha}{2} - 2} \{-y^2 \cos[\alpha(\theta - \omega_1)] + xy \sin[\alpha(\theta - \omega_1)]\} \\ &+ \alpha(x^2 + y^2)^{\frac{\alpha}{2} - 1} \cos[\alpha(\theta - \omega_1)] \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} |\psi_{xy}(x, y)| &\leqslant (x^2 + y^2)^{\frac{\alpha}{2} - 2} \{\alpha(2 - \alpha)(|xy| + x^2) + \alpha^2 (y^2 + |xy|)\} \\ &\quad + \alpha(x^2 + y^2)^{\frac{\alpha}{2} - 1} \\ &\leqslant \alpha(\alpha + 4)\, (x^2 + y^2)^{\frac{\alpha}{2} - 1}, \qquad \forall (x, y) \in \Omega. \end{aligned} \tag{2.14}$$

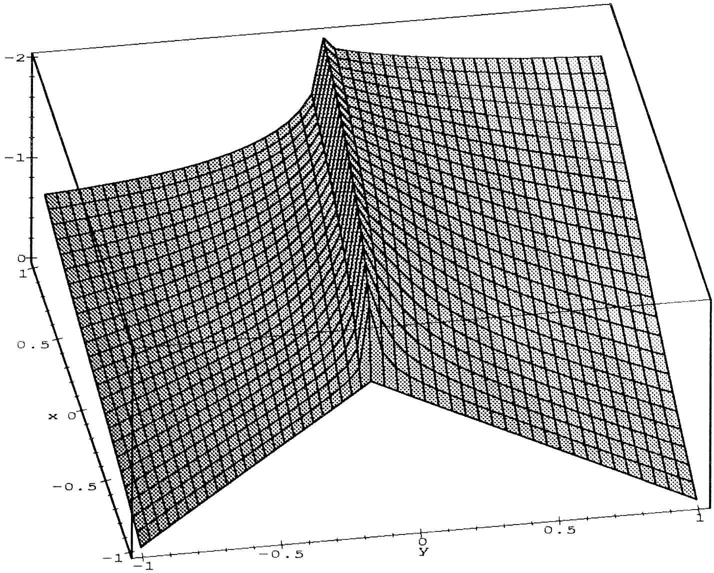

FIG. 1. The picture for $p_s = -\mathbf{C} \cdot B \nabla\psi$ with $\mathbf{C} = [1, 1]$.

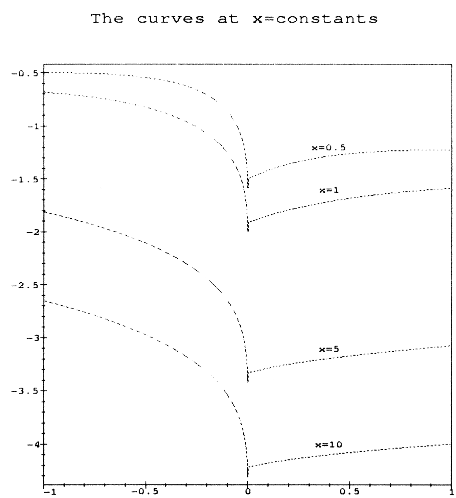

FIG. 2. The curves for p_s at x=constants.

In a similar manner, the partial derivatives $\psi_{xx}(x, y)$ and $\psi_{yy}(x, y)$ have the same bounds given in (2.14) and

$$|\nabla\psi_y(x, y)| \leqslant c_\alpha (x^2+y^2)^{\frac{\alpha}{2}-1}, \tag{2.15}$$

where $c_\alpha = \sqrt{2}\,\alpha(\alpha+4)$. In order to compute the L^2-norm of the function p_s defined in (2.8) we first estimate

$$\begin{aligned}
|p_s(x, y)| &\leqslant |\mathbf{C}| \left| \int_{\delta(y)}^{x} \nabla\phi(s, y)\, ds \right| \\
&\leqslant |\mathbf{C}| \left(\int_{\delta(y)}^{x} |\nabla\psi(s, y)|\, ds + |\nabla\chi|_\infty \int_{\delta(y)}^{x} |\psi(s, y)|\, ds \right) \\
&\qquad \text{(using (2.12) and letting } c_1 = |\nabla\chi|_\infty) \\
&\leqslant |\mathbf{C}| \left(c_\alpha \int_{\delta(y)}^{x} (s^2+y^2)^{\frac{\alpha-1}{2}}\, ds + c_1 \int_{\delta(y)}^{x} (s^2+y^2)^{\frac{\alpha}{2}}\, ds \right) \\
&\qquad \left(\text{letting } t = \frac{s}{y} \text{ and } \delta(y) = my \right) \\
&\leqslant |\mathbf{C}| \left(c_\alpha y^\alpha \int_m^{\frac{x}{y}} (t^2+1)^{\frac{\alpha-1}{2}}\, dt + c_1 y^{\alpha+1} \int_m^{\frac{x}{y}} (t^2+1)^{\frac{\alpha}{2}}\, dt \right) \\
&\leqslant k_1\, |\mathbf{C}|\, (x^2+y^2)^{\frac{\alpha}{2}},
\end{aligned} \tag{2.16}$$

where $k_1 = 4 \max\{1+|m|^\alpha, 1+\alpha\,|m|, c_1[1+(m^2+1)^{\frac{\alpha}{2}}]\}$. Indeed, for $m > 0$

$$\int_m^{\frac{x}{y}} (t^2+1)^{\frac{\alpha-1}{2}}\, dt < \int_m^{\frac{x}{y}} t^{\alpha-1}\, dt = \frac{1}{\alpha}\left[\left(\frac{x}{y}\right)^\alpha - m^\alpha\right] \tag{2.17}$$

and for $m < 0$

$$\begin{aligned}
\int_m^{\frac{x}{y}} (t^2+1)^{\frac{\alpha-1}{2}}\, dt &= \int_m^0 (t^2+1)^{\frac{\alpha-1}{2}}\, dt + \int_0^{\frac{x}{y}} (t^2+1)^{\frac{\alpha-1}{2}}\, dt \\
&\leqslant |m| + \frac{1}{\alpha}\left(\frac{x}{y}\right)^\alpha.
\end{aligned} \tag{2.18}$$

Hence it follows from (2.16) that

$$\begin{aligned}
\|p_s\|_{0,\Omega} &\leqslant k_1\, |\mathbf{C}| \left(\int_{\theta=\omega_1}^{\omega_2} \int_0^1 r^{2\alpha} r\, dr\, d\theta \right)^{\frac{1}{2}} \\
&= k_2\, |\mathbf{C}|
\end{aligned} \tag{2.19}$$

where $k_2 = k_1\sqrt{\omega/(2(1+\alpha))}$. Next, to estimate $\|\nabla p_s\|_{0,\Omega}$, we differentiate both sides of (2.8) with respect to y and get

$$p_{s,y}(x, y) = -\mathbf{C}\cdot \int_{\delta(y)}^{x} \nabla\phi_y(s, y)\,ds + \mathbf{C}\cdot\nabla\phi(\delta(y), y). \tag{2.20}$$

Hence it follows from (2.12) and (2.15) that

$$\begin{aligned}
|p_{s,y}(x, y)| &\leqslant |\mathbf{C}|\left(\int_{\delta(y)}^{x} |\nabla\psi_y(s, y)| + 2c_1\,|\nabla\psi(s, y)| + c_2\,|\psi(s, y)|\,ds \right.\\
&\quad \left. + |\nabla\psi(\delta(y), y)| + c_1\,|\psi(\delta(y), y)|\right) \qquad (2.21)\\
&\quad\quad (\text{letting } c_2 = 2\,\|\chi\|_{2,\infty} \text{ and } c_3 = (2\alpha + c_1)(1+m^2)^{\frac{\alpha}{2}})\\
&\leqslant |\mathbf{C}|\left(c_\alpha \int_{\delta(y)}^{x} (s^2+y^2)^{\frac{\alpha}{2}-1}\,ds + c_2\int_{\delta(y)}^{x} (s^2+y^2)^{\frac{\alpha-1}{2}} + (s^2+y^2)^{\frac{\alpha}{2}}\,ds \right.\\
&\quad \left. + c_3\, y^{\alpha-1}(1+y)\right)\\
&\quad\quad \left(\text{noting that } \delta(y) = m\,y \text{ and letting } t = \frac{s}{y} \text{ and } ds = y\,dt\right)\\
&\leqslant |\mathbf{C}|\left(c_\alpha y^{\alpha-1}\int_{m}^{\frac{x}{y}} (t^2+1)^{\frac{\alpha}{2}-1}\,ds + c_4(x^2+y^2)^{\frac{\alpha}{2}} + c_3 y^{\alpha-1}(1+y)\right)\\
&\leqslant |\mathbf{C}|\,c_\alpha y^{\alpha-1}\left(\int_{m}^{1} (t^2+1)^{\frac{\alpha}{2}-1}\,dt + \int_{1}^{\frac{x}{y}} t^{\alpha-2}\,dt\right) \qquad (2.22)\\
&\quad + |\mathbf{C}|\,(c_4(x^2+y^2)^{\frac{\alpha}{2}} + c_3 y^{\alpha-1}(1+y)) \qquad (\text{say } m<0)\\
&\leqslant |\mathbf{C}|\,(c_3 y^{\alpha-1}(1+y) + c_4(x^2+y^2)^{\frac{\alpha}{2}} + c_5(x^2+y^2)^{\frac{\alpha-1}{2}}), \qquad (2.23)
\end{aligned}$$

where $c_4 = 2\,c_2(1+|m|+\alpha^{-1})$ and $c_5 = \sqrt{2}\,c_\alpha\,((1-\alpha)^{-1}+1-m)$. The integral term of (2.22) is finite since, for $m>0$

$$\int_{m}^{\infty} (t^2+1)^{\frac{\alpha}{2}-1}\,dt \leqslant \int_{m}^{\infty} t^{\alpha-2}\,dt = \frac{m^{\alpha-1}}{1-\alpha}. \tag{2.24}$$

Hence using (2.23) the L^2-norm of $p_{s,y}$ is estimated by

$$\begin{aligned}\|p_{s,y}\|_{0,\Omega} &\leqslant 3\,|\mathbf{C}| \left(c_3 \sqrt{\frac{4\alpha}{4\alpha^2-1}} + c_4 \sqrt{\frac{\omega}{\alpha+2}} + c_5 \sqrt{\frac{\omega}{\alpha}} \right) \\ &= c_6\,|\mathbf{C}|, \end{aligned} \tag{2.25}$$

where $c_6 = 3(c_3 \sqrt{4\alpha/(4\alpha^2-1)} + c_4 \sqrt{\omega/(\alpha+2)} + c_5 \sqrt{\omega/\alpha}\,)$. Using (2.12) we also compute the L^2 norm of $p_{s,x}$

$$\begin{aligned}\|p_{s,x}\|_{0,\Omega} &\leqslant |\mathbf{C}|\,(\|\nabla\psi\|_{0,\Omega} + c_1\,\|\psi\|_{0,\Omega}) \\ &\leqslant \sqrt{2}\,|\mathbf{C}| \left(2\alpha \sqrt{\frac{\omega}{\alpha+1}} + c_1 \sqrt{\frac{\omega}{\alpha+2}} \right) \\ &\leqslant c_7\,|\mathbf{C}|, \end{aligned} \tag{2.26}$$

where $c_7 = \sqrt{2}\,(2\alpha \sqrt{\omega/(\alpha+1)} + c_1 \sqrt{\omega/(\alpha+2)}\,)$. Thus, combining (2.19) and (2.25)–(2.26) the inequality (2.9) follows with the constant $K_1 = \max\{k_2, c_6, c_7\}$. ∎

Since $\mathbf{U}\cdot\nabla p_s + \operatorname{div}\mathbf{u}_s = 0$ in Ω and $p = p_r + p_s$, $[\mathbf{u}_r, p]$ is the solution of the problem

$$\begin{cases} -\mu\,\Delta\mathbf{u}_r + \nabla p = \mathbf{f} + \mathbf{f}_0 & \text{in } \Omega, \\ \mathbf{U}\cdot\nabla p + \operatorname{div}\mathbf{u}_r = g - \operatorname{div}\mathbf{u}_s & \text{in } \Omega, \\ \mathbf{u}_r = 0 & \text{on } \Gamma, \\ p = 0 & \text{on } \Gamma_{in}, \end{cases} \tag{2.27}$$

where $\mathbf{f}_0 := \mu\,\mathbf{C}\,\Delta\phi = \mu\,\mathbf{C}(\psi\,\Delta\chi + 2\nabla\chi\cdot\nabla\psi)$. Consequently, if $[\mathbf{u}_r, p]$ is the solution of (2.27) with $\mathbf{u}_r \in (H^2(\Omega))^2$ and $g \in H^1(\Omega)$, Lemma 2.2 yields

$$\begin{aligned}\|p\|_{1,\Omega} &\leqslant \|B(g - \operatorname{div}\mathbf{u}_r)\|_{1,\Omega} + \|B\operatorname{div}\mathbf{u}_s\|_{1,\Omega} \\ &\leqslant C(\|g\|_{1,\Omega} + \|\mathbf{u}_r\|_{2,\Omega}) + K_1\,|\mathbf{C}|,\end{aligned}$$

where K_1 is given in (2.9). That is to say, we have $p \in H^1(\Omega)$. Shortly, we will show how to select $\mathbf{C}$ so as to insure that $\mathbf{u}_r \in (H^2(\Omega))^2$.

We use the solution operators A and B to express the solution $[\mathbf{u}_r, p]$ of (2.27) as

$$\begin{aligned}\mathbf{u}_r &= \frac{1}{\mu} A(\mathbf{f} + \mathbf{f}_0 - \nabla p) \in (H^2(\Omega))^2, \\ p &= B(g - \operatorname{div}\mathbf{u}_r - \operatorname{div}\mathbf{u}_s) \in H^1(\Omega).\end{aligned} \tag{2.28}$$

At this point, from the momentum equation (2.27a) and the point of view of Theorem 2.1 we must pick the unknown constant $\mathbf{C}=[C_1, C_2]$ appearing in the function $\mathbf{u}_s$ of (2.6) so that

$$\begin{cases} \dfrac{1}{\mu} A(f_1+f_{01}-p_x)=0, \\ \dfrac{1}{\mu} A(f_2+f_{02}-p_y)=0. \end{cases} \tag{2.29}$$

Applying the divergence operator to the function $\mathbf{u}_r$ given in (2.28a) and inserting it into (2.28b), the solution p may be expressed in the form

$$\begin{aligned} p &= B\left(g-\frac{1}{\mu}\operatorname{div}\left[A(\mathbf{f}+\mathbf{f}_0-\nabla p)\right]-\operatorname{div}\mathbf{u}_s\right) \\ &= Bg-\frac{1}{\mu}B\operatorname{div}\left[A(\mathbf{f}+\mathbf{f}_0)\right]+\frac{1}{\mu}B\operatorname{div}(A\,\nabla p)-B\operatorname{div}\mathbf{u}_s. \end{aligned} \tag{2.30}$$

Noting that $B\operatorname{div}\mathbf{u}_s=\mathbf{C}\cdot B\,\nabla\phi$, we have

$$\begin{aligned} \left[I-\frac{1}{\mu}B\operatorname{div}(A\,\nabla)\right]p &= Bg-\frac{1}{\mu}B\operatorname{div}\left[A(\mathbf{f}+\mathbf{f}_0)\right]-\mathbf{C}\cdot B\,\nabla\phi \\ &= Bg-\frac{1}{\mu}B\operatorname{div}A\mathbf{f}-B\operatorname{div}A(\mathbf{C}\,\Delta\phi)-\mathbf{C}\cdot B\,\nabla\phi \\ &= Bg-\frac{1}{\mu}B\operatorname{div}A\mathbf{f}-\mathbf{C}\cdot\beta, \end{aligned} \tag{2.31}$$

where $\beta=[\beta_1, \beta_2]$ is a vector with components

$$\beta_1 = B\,\nabla_x A\,\Delta\,\phi - B\,\phi_x$$
$$\beta_2 = B\,\nabla_y A\,\Delta\,\phi - B\,\phi_y.$$

Hence, if μ is sufficiently large, the operator

$$B^* := \left[I-\frac{1}{\mu}B\operatorname{div}(A\,\nabla)\right]^{-1} \tag{2.32}$$

will exist (to be shown in Lemma 2.3) and the function p can be solved explicitly:

$$p=-\mathbf{C}\cdot B^*\beta+B^*\left(Bg-\frac{1}{\mu}B\operatorname{div}(A\mathbf{f})\right). \tag{2.33}$$

On the other hand, substituting the function p of (2.28b) into the equation (2.28a) we have

$$u_r = \frac{1}{\mu} A[f_1 + f_{01} - \nabla_x B(g - \operatorname{div} \mathbf{u}_r - \operatorname{div} \mathbf{u}_s)]$$

(noting that $\nabla_x BG = G$)

$$= \frac{1}{\mu} A[f_1 + f_{01} - g + \operatorname{div} \mathbf{u}_r + \operatorname{div} \mathbf{u}_s]$$

(noting that $f_{01} = \mu\, C_1\, \Delta\phi$)

$$= \frac{1}{\mu} A\, \nabla_x B \operatorname{div} \mathbf{u}_r + \left(\frac{1}{\mu} A\, \nabla_x B\, \nabla\phi + A\, \Delta\phi\right) C_1 + \left(\frac{1}{\mu} A\, \nabla_x B\, \nabla\phi\right) C_2$$
$$+ \frac{1}{\mu} A(f_1 - g), \tag{2.34}$$

$$v_r = \frac{1}{\mu} A[f_2 + f_{02} - \nabla_y B(g - \operatorname{div} \mathbf{u}_r - \operatorname{div} \mathbf{u}_s)]$$

(noting that $f_{02} = \mu\, C_2\, \Delta\phi$)

$$= \frac{1}{\mu} A\, \nabla_y B \operatorname{div} \mathbf{u}_r + \left(\frac{1}{\mu} A\, \nabla_y B\, \phi\right) C_1 + \left(\frac{1}{\mu} A\, \nabla_y B\, \nabla\phi + A\, \Delta\phi\right) C_2$$
$$+ \frac{1}{\mu} A(f_2 - \nabla_y Bg). \tag{2.35}$$

Hence using (2.34)–(2.35) the function $\mathbf{u}_r = [u_r, v_r]$ can be written by

$$\mathbf{u}_r = \frac{1}{\mu} A\, \nabla B \operatorname{div} \mathbf{u}_r + \alpha \cdot \mathbf{C} + \frac{1}{\mu} A(\mathbf{f} - \nabla Bg), \tag{2.36}$$

where

$$\alpha = \begin{pmatrix} \frac{1}{\mu} A\, \nabla_x B\, \nabla\phi + A\, \Delta\phi & \frac{1}{\mu} A\, \nabla_x B\, \nabla\phi \\ \frac{1}{\mu} A\, \nabla_y B\, \nabla\phi & \frac{1}{\mu} A\, \nabla_y B\, \nabla\phi + A\Delta\phi \end{pmatrix} \tag{2.37}$$

and

$$\left(I - \frac{1}{\mu} A\, \nabla B \operatorname{div}\right) \mathbf{u}_r = \alpha \cdot \mathbf{C} + \frac{1}{\mu} A(\mathbf{f} - \nabla Bg). \tag{2.38}$$

Thus, if μ is large enough then the operator

$$A^* := \left(I - \frac{1}{\mu} A \, \nabla B \, \mathrm{div} \right)^{-1} \tag{2.39}$$

exists (to be shown in Lemma 2.3) and the function $\mathbf{u}_r$ can be solved explicitly:

$$\mathbf{u}_r = A^* \alpha \cdot \mathbf{C} + \frac{1}{\mu} A^* A (\mathbf{f} - \nabla B g). \tag{2.40}$$

Consequently, if we know the constant $\mathbf{C} = [C_1, C_2]$ then the solution $[\mathbf{u}_r, p]$ of (2.27) can be explicitly represented.

LEMMA 2.3. (a) *Let A and B be the solution operators defined in* (1.11) *and* (1.12), *respectively. Then the following norms are bounded:*

$$\| A \, \nabla B \, \mathrm{div} \, \| := \sup_{0 \neq \mathbf{v} \in (H^1(\Omega))^2} \frac{\| A \, \nabla B \, \mathrm{div} \, \mathbf{v} \|_{1, \Omega}}{\| \mathbf{v} \|_{1, \Omega}} < \infty, \tag{2.41}$$

$$\| B \, \mathrm{div} \, (A \, \nabla) \| := \sup_{0 \neq \chi \in L^2(\Omega)} \frac{\| B \, \mathrm{div} (A \, \nabla \chi) \|_{0, \Omega}}{\| \chi \|_{0, \Omega}} < \infty. \tag{2.42}$$

(b) *Assume that μ is large enough. Then the operators A^* and B^* defined in* (2.39) *and* (2.32), *respectively exist and the solution $[\mathbf{u}_r, p]$ of* (2.27) *is given by*

$$\begin{cases} \mathbf{u}_r = A^* \alpha \cdot \mathbf{C} + \dfrac{1}{\mu} A^* A (\mathbf{f} - \nabla B g), \\ p = -B^* \beta \cdot \mathbf{C} + B^* \left(B g - \dfrac{1}{\mu} B \, \mathrm{div} \, (A \mathbf{f}) \right), \end{cases} \tag{2.43}$$

where α and β are given in (2.37) *and* (2.32), *respectively.*

Proof. First, (2.41)–(2.42) follow from the following two diagrams:

$$(H^1)^2 \xrightarrow{\mathrm{div}} L^2 \xrightarrow{B} Q \xrightarrow{\nabla} (H^{-1})^2 \xrightarrow{A} (H^1)^2, \tag{2.44}$$

$$L^2 \xrightarrow{\nabla} (H^{-1})^2 \xrightarrow{A} (H^1)^2 \xrightarrow{\mathrm{div}} L^2 \xrightarrow{B} Q \subset L^2. \tag{2.45}$$

In order to prove (b), let $z \in L^2(\Omega)$ be arbitrary. Then we have

$$\left\| \left[I - \frac{1}{\mu} B \operatorname{div}(A \nabla) \right] z \right\|_{0,\Omega} \geqslant \left(1 - \frac{1}{\mu} \|B \operatorname{div}(A \nabla)\| \right) \|z\|_{0,\Omega}.$$

Hence if μ is large enough then B^* is bounded on $L^2(\Omega)$. Similarly the operator A^* is also bounded on H^1. ∎

We now go back to Eqs. (2.29). First, taking the derivatives of p in (2.33) we have

$$\nabla_x p = -\nabla_x B^* \beta \cdot \mathbf{C} + \nabla_x B^* \left(Bg - \frac{1}{\mu} B \operatorname{div} A\mathbf{f} \right), \tag{2.46}$$

$$\nabla_y p = -\nabla_y B^* \beta \cdot \mathbf{C} + \nabla_y B^* \left(Bg - \frac{1}{\mu} B \operatorname{div} A\mathbf{f} \right). \tag{2.47}$$

Hence, using (2.29), (2.46)–(2.47) and $f_{01} = \mu C_1 \Delta\phi$, $f_{02} = \mu C_2 \Delta\phi$ we obtain algebraic equations for the unknown parameter $\mathbf{C} = [C_1, C_2]$:

$$\begin{cases} \lambda_{11} C_1 + \lambda_{12} C_2 = M_1 \\ \lambda_{21} C_1 + \lambda_{22} C_2 = M_2, \end{cases} \tag{2.48}$$

where

$$\begin{cases} \lambda_{11} = \mu \Lambda(\Delta\phi) + \Lambda(\nabla_x B^* \beta_1) \\ \lambda_{12} = \Lambda(\nabla_x B^* \beta_2) \\ \lambda_{21} = \Lambda(\nabla_y B^* \beta_1) \\ \lambda_{22} = \mu \Lambda(\Delta\phi) + \Lambda(\nabla_y B^* \beta_2) \\ M_1 = \Lambda(\nabla_x B^* B g) - \dfrac{1}{\mu} \Lambda(\nabla_x B^* B \operatorname{div} A\mathbf{f}) - \Lambda(f_1) \\ M_2 = \Lambda(\nabla_y B^* B g) - \dfrac{1}{\mu} \Lambda(\nabla_y B^* B \operatorname{div} A\mathbf{f}) - \Lambda(f_2). \end{cases} \tag{2.49}$$

In order to show that the coefficients given in (2.49) are well-defined it is enough to show the following lemma.

LEMMA 2.4. *Let* $\frac{1}{2} < \alpha = \frac{\pi}{\omega} < 1$ *and let* $\phi(x, y)$ *be given in* (2.2). *Then the following functions:* $(\nabla B^* B \nabla A \Delta) \phi$ *and* $(\nabla B^* B \nabla) \phi$ *belong to* $L^2(\Omega)$. *Furthermore* $\Lambda(\nabla B^* B \nabla A \Delta\phi)$, $\Lambda(\nabla B^* B \nabla) \phi$, *and* $\Lambda(\Delta\phi)$ *will be finite where* Λ *is defined in* (2.3).

Proof. From (2.2) we have $\phi = \chi \psi$ with $\psi(x, y) = r^{\alpha} \sin[\alpha (\theta - \omega_1)]$ where θ and ω_1 are defined in (2.1). Hence from (2.12) we have $|\nabla \phi(x, y)| \leqslant 2 \alpha r^{\alpha - 1}$ for all $(x, y) \in \Omega$. From the procedures given in the proofs of (2.16) and (2.21)–(2.23) one may observe the following inequalities:

$$|B \nabla \phi(x, y)| = \left| \int_{\delta(y)}^{x} \nabla \phi(s, y) \, ds \right| \leqslant C r^{\alpha}, \tag{2.50}$$

and

$$\begin{aligned} |\nabla_y B \nabla \phi(x, y)| &\leqslant \int_{\delta(y)}^{x} |\nabla \phi_y(s, y)| \, ds + |\nabla \phi(\delta(y), y)| \, |\delta'(y)| \\ &\leqslant C r^{\alpha - 1} \end{aligned} \tag{2.51}$$

for all $(x, y) \in \Omega$ where C is a positive constant. In other words, we have $B \nabla \phi \in H^1(\Omega)$ and $\nabla_y B \nabla \phi \in L^2(\Omega)$. Since B^* is a bounded operator by Lemma 2.3, $(\nabla B^* B \nabla) \phi \in L^2(\Omega)$. Since $\Delta \phi \equiv 0$ near the origin we see that $(\nabla B^* B \nabla \Lambda \Delta) \phi \in L^2(\Omega)$. Since the linear functional Λ is bounded on $L^2(\Omega)$ the required results easily follows. ▮

Thus, using Lemma 2.4 and recalling that $\beta = B \nabla \Lambda \Delta \phi - B \nabla \phi$, we can conclude that the coefficients in (2.49) are well-defined.

LEMMA 2.5. (a) *The numbers* λ_{ij} *defined in* (2.49) *are finite* $(i, j = 1, 2)$ *and also if* $\mathbf{f} \in L^2$ *and* $g \in L^2$ *then* M_1 *and* M_2 *are finite and estimated by* $C(\|\mathbf{f}\|_{0, \Omega} + \|g\|_{1, \Omega})$. (b) *If* μ *is large enough, the determinant* $\lambda_{11} \lambda_{22} - \lambda_{12} \lambda_{21}$ *is not zero.*

Proof. (a) The boundedness of the numbers λ_{ij} follows from Lemma 2.4. For the computation of the right hand sides of M_i (i$=1$, 2) we first estimate

$$\begin{aligned} |\Lambda(\nabla B^* B g)| &\leqslant C \, \|\nabla B^* B g\|_{0, \Omega} \qquad \text{(since } B^* \text{ is a bounded operator)} \\ &\leqslant C \, \|B^*\| \, \|\nabla B g\|_{0, \Omega} \\ &\leqslant C \, \|B^*\| \, \|B\| \, \|\nabla g\|_{0, \Omega}, \end{aligned}$$

where C is a constant. From the following diagram

$$(H^{-1})^2 \xrightarrow{\Lambda} (H^1)^2 \xrightarrow{\text{div}} L^2 \xrightarrow{B} Q \xrightarrow{B^*} Q \xrightarrow{\nabla} (H^{-1})^2$$

the following norm will be bounded:

$$\|\nabla B^*B \operatorname{div} A\| = \sup_{0 \neq v \in (H^{-1})^2} \frac{\|\nabla B^*B \operatorname{div} Av\|_{-1,\Omega}}{\|v\|_{-1,\Omega}}$$

and

$$\begin{aligned}|\Lambda(\nabla B^*B \operatorname{div} A\mathbf{f})| &\leqslant C\,\|\nabla B^*B \operatorname{div} A\mathbf{f}\|_{-1,\Omega}\\ &\leqslant C\,\|\nabla B^*B \operatorname{div} A\|\,\|f\|_{-1,\Omega}\\ &\leqslant C\,\|\nabla B^*B \operatorname{div} A\|\,\|f\|_{0,\Omega}.\end{aligned}$$

Hence it follows from the above inequalities that $|M_i| \leqslant C(\|\mathbf{f}\|_{0,\Omega} + \|g\|_{1,\Omega})$ (i=1, 2) where C is a constant.

(b) If μ is large enough, the number $\lambda_{11}\lambda_{22}$ will be greater than $\lambda_{12}\lambda_{21}$ (see (2.49)) and the determinant will not be zero. ∎

Now the constant parameter $\mathbf{C}$ in (2.6) can be determined so that the solution $[\mathbf{u}, p]$ of (1.8) may be split into a singular and regular part. Hence using the algebraic equation (2.48) and Lemma 2.5 we deduce the following:

THEOREM 2.2. *If μ is sufficiently large then the constant unknown $\mathbf{C} = [C_1, C_2]$ in* (2.6) *is determined so that*

$$\begin{aligned}C_1 &= (M_1\lambda_{22} - M_2\lambda_{12})/(\lambda_{11}\lambda_{22} - \lambda_{21}\lambda_{12})\\ C_2 &= (M_2\lambda_{11} - M_1\lambda_{21})/(\lambda_{11}\lambda_{22} - \lambda_{21}\lambda_{12}).\end{aligned} \tag{2.52}$$

Morever it is estimated by the norms of the data $\mathbf{f}$ *and* g,

$$|\mathbf{C}| \leqslant C(\|\mathbf{f}\|_{0,\Omega} + \|g\|_{1,\Omega}), \tag{2.53}$$

where $C = C(\lambda_{ij})$ $(i, j=1, 2)$.

We are now ready to obtain an a priori estimate for the solution $[\mathbf{u}_r, p]$.

THEOREM 2.3. *Suppose that the hypotheses of Theorem* 1.4 *hold. Assume that the constant* $\mathbf{C} = [C_1, C_2]$ *has been determined in Theorem* 2.2 *so that* (2.29) *holds. Then if* $\mu > \|A\|\,(\|B\|+1)$, $[\mathbf{u}_r, p]$ *satisfies the inequality*

$$\|\mathbf{u}_r\|_{2,\Omega} + \|p\|_{1,\Omega} \leqslant C(\|\mathbf{f}\|_0 + \|g\|_{1,\Omega}). \tag{2.54}$$

Proof. To get the inequality (2.54), we first consider the continuity equation (2.27b). Then using the operator B, the solution p is given by

$$p = BG, \tag{2.55}$$

where $G := g - \operatorname{div} \mathbf{u}_r - \operatorname{div} \mathbf{u}_s$. Using Lemma 2.2, we have

$$\begin{aligned} \|p\|_{1,\Omega} &\leqslant (\|B\|+1)(\|\operatorname{div} \mathbf{u}_r\|_{1,\Omega} + \|g\|_{1,\Omega}) + K_1\ |\mathbf{C}| \qquad \text{(using (2.53))} \\ &\leqslant (\|B\|+1)\ \|\mathbf{u}_r\|_{2,\Omega} + C(\|\mathbf{f}\|_{0,\Omega} + \|g\|_{1,\Omega}), \end{aligned} \tag{2.56}$$

where $\|B\|$ is the norm of operator B and K_1 is given in (2.9). Second, applying to Theorem 2.1 the solution $\mathbf{u}_r := \mathbf{u} - \mathbf{u}_s$ for the momentum equation of (2.27): $-\mu\,\Delta\mathbf{u}_r = F$ in Ω, $\mathbf{u}_r = 0$ on Γ where $F := -\nabla p + \mathbf{f} + \mathbf{f}_0$, we have

$$\begin{aligned} \mu\ \|\mathbf{u}_r\|_{2,\Omega} &\leqslant \|A\|\ (\|\nabla p\|_{0,\Omega} + \|\mathbf{f}_0\|_{0,\Omega} + \|\mathbf{f}\|_{0,\Omega}) \\ &\qquad \text{(using (2.56) and } \mathbf{f}_0 = \mu\,\mathbf{C}\,\Delta\phi\,) \\ &\leqslant \|A\|\ (\|B\|+1)\ \|\mathbf{u}_r\|_{2,\Omega} + \mu\ \|A\|\ \|\Delta\phi\|_{0,\Omega}\ |\mathbf{C}| + C(\|\mathbf{f}\|_{0,\Omega} + \|g\|_{1,\Omega}) \\ &\qquad \text{(using (2.53))} \\ &\leqslant \|A\|\ (\|B\|+1)\ \|\mathbf{u}_r\|_{2,\Omega} + C(\|\mathbf{f}\|_{0,\Omega} + \|g\|_{1,\Omega}), \end{aligned} \tag{2.57}$$

where $C = C(\mu, \alpha, \omega, \|A\|, \|B\|)$. Thus, if $\mu > \|A\|\ (\|B\|+1)$ then

$$\|\mathbf{u}_r\|_{2,\Omega} \leqslant C(\|\mathbf{f}\|_{0,\Omega} + \|g\|_{1,\Omega}), \tag{2.58}$$

where $C = C(\mu, \alpha, \omega, \|A\|, \|B\|)$. Thus combining (2.56) and (2.58) yields the required inequality (2.54). ∎

In summary, combining (2.6), Theorem 2.2 and Theorem 2.3 we have obtained the following result:

THEOREM 2.4. *Let Ω be the concave sector defined in* (2.1). *Let* $\mathbf{f} \in (L^2(\Omega))^2$, $g \in H^1(\Omega)$ *and let* $[\mathbf{u}, p] \in (H^1(\Omega) \cap H^2_{\mathrm{loc}}(\Omega))^2 \times (L^2(\Omega) \cap H^1_{\mathrm{loc}}(\Omega))$ *be the solution of* (1.8) *in both the weak and pointwise sense with* $\mathbf{u} \equiv 0$, $p \equiv 0$ *for* $r \geqslant 1$. *Then the velocity* $\mathbf{u}$ *is split into the singular and regular parts,* $\mathbf{u}_s$ *and* $\mathbf{u}_r$ *near the origin: if* ω *is the angle of the concave vertex of* (2.1) *so that* $\alpha := \frac{\pi}{\omega} < 1$, *and* χ *is the smooth cutoff function,*

$$\mathbf{u} = \mathbf{u}_s + \mathbf{u}_r, \qquad \mathbf{u}_s = \mathbf{C}\phi \text{ with } \phi(x, y) = \chi(r)\, r^\alpha \sin[\alpha(\theta - \omega_1)] \tag{2.59}$$

and $[\mathbf{u}_r, p] \in (H^2(\Omega))^2 \times H^1(\Omega)$ *where* $\mathbf{C}$ *is the constant of* (2.52). *Furthermore there is a constant* $C = C(D, \mu)$ *such that if* μ *is large properly,*

$$\|\mathbf{u}_r\|_{2,\Omega} + \|p\|_{1,\Omega} \leqslant C(\|\mathbf{f}\|_{0,\Omega} + \|g\|_{1,\Omega}). \tag{2.60}$$

We are now ready to obtain Theorem 1.4. Note that Theorem 2.4 was shown for the concave sector defined in (2.1). Except for the sector (2.1), other cases of concave sectors are the following:

$$\Omega = \{(x, y) \mid r = \sqrt{x^2 + y^2}, \theta = \tan^{-1}\frac{y}{x}, \omega_1 < \theta < \omega_2, 0 < r < \infty\}, \tag{2.61}$$

where $0 < \omega_1 < \frac{\pi}{2}, \frac{3\pi}{2} < \omega_2 \leqslant 2\pi$. In this case the incoming portion $\Gamma_{in} = \varnothing$.

$$\Omega = \{(x, y) \mid 0 \leqslant \theta < \omega_1, \omega_2 < \theta < 2\pi, 0 < r < \infty\}, \tag{2.62}$$

where $0 < \omega_1 < \frac{\pi}{2}, \frac{\pi}{2} < \omega_2 \leqslant \pi$. In this case $\Gamma_{in} = \{(x, y) \in \Gamma : \theta = \omega_2\}$.

$$\Omega = \{(x, y) \mid 0 \leqslant \theta < -\omega_1, -\omega_2 < \theta < 2\pi, 0 < r < \infty\}, \tag{2.63}$$

where $\omega_1 \in (-\frac{3\pi}{2}, -\pi)$ and $\omega_2 \in (-2\pi, -\frac{3\pi}{2})$. In this case $\Gamma_{in} = \{(x, y) \in \Gamma : \theta = \omega_1\}$. Now, for the concave sectors defined in (2.61)–(2.63), the same procedures given in this section may be applied and the same conclusions can be drawn. Thus Theorem 1.4 follows.

3. CONVEX SECTOR

In this section Theorem 1.5 is proven. Before proceeding further we cite the following result (see [3]):

THEOREM 3.1. *If* Ω *is a convex polygon and* $F \in L^2(\Omega)$, *the solution* u *of the problem* $-\Delta u = F$ *in* Ω, $u = 0$ *on* Γ, *has square integrable second derivatives, and* $\|u\|_{2,\Omega} \leqslant C\|F\|_{0,\Omega}$ *for some constant* $C = C(\Omega)$.

We consider problem (1.8) on the convex sector

$$\Omega = \{(x, y) \mid r = \sqrt{x^2 + y^2}, \theta = \tan^{-1}\frac{y}{x}, 0 < r < \infty, \omega_1 < \theta < \omega_2\}, \tag{3.1}$$

where $-2\pi < \omega_1 \leqslant -\frac{3\pi}{2}, 0 < \omega_2 \leqslant \pi/2$. For the other case (see (3.12)) of the convex sector it is not difficult to show the results to be given in this section by following the similar procedures used. Also, for the other cases except for the convex sectors of (3.1) and (3.12) the same procedures used can applied.

For the regularity result of the solution of (1.8) on the domain Ω in (3.1) we first consider the following first order partial differential equation:

$$\begin{cases} p_x = G & \text{in } \Omega \\ p = 0 & \text{on } \Gamma_{in}, \end{cases} \tag{3.2}$$

where G is a given function in Ω. Hence the solution p of (3.2) is given by

$$p(x, y) = \begin{cases} \displaystyle\int_{\delta_+(y)}^{x} G(s, y)\, ds := p^+(x, y) & (y > 0) \\ \displaystyle\int_{\delta_-(y)}^{x} G(s, y)\, ds := p^-(x, y) & (y < 0), \end{cases} \tag{3.3}$$

where $(\delta_+(y), y)$ and $(\delta_-(y), y)$ describe the incoming portions of the boundary Γ, $\Gamma_{in} \cap \{y > 0\}$ and $\Gamma_{in} \cap \{y < 0\}$, respectively. Note that the two functions $\delta_+(y)$ and $\delta_-(y)$ go to zero as y approaches zero. We observe that the functions $p^+(x, y)$ and $p^-(x, y)$ will converge to the same limit

$$p(x, 0) = \int_0^x G(s, 0)\, ds \tag{3.4}$$

as y goes to zero. For example, if $G = 1$ then $p(x, 0) = x$. On the other hand, taking the derivative to $p(x, y)$ given in (3.3) with respect to the variable y,

$$p_y(x, y) = \begin{cases} \displaystyle\int_{\delta_+(y)}^{x} G_y(s, y)\, ds - G(\delta_+(y), y)\, \delta'_+(y) & (y > 0) \\ \displaystyle\int_{\delta_-(y)}^{x} G_y(s, y)\, ds - G(\delta_-(y), y)\, \delta'_-(y) & (y < 0). \end{cases} \tag{3.5}$$

Hence $p_y(x, y)$ converges to the different limits as $y \to \pm 0$:

$$\begin{cases} p_y(x, +0) = \displaystyle\int_0^x G_y(s, 0)\, ds - G(0, 0)\, \delta'_+(0) & (y > 0) \\ p_y(x, -0) = \displaystyle\int_0^x G_y(s, 0)\, ds - G(0, 0)\, \delta'_-(0) & (y < 0) \end{cases} \tag{3.6}$$

and the value of the jump in $p_y(x, y)$ across $y=0$ is

$$p_y(x,+0)-p_y(x,-0)=-G(0,0)[\delta'_+(0)-\delta'_-(0)]. \tag{3.7}$$

Since the p_y is not continuous arcoss $y=0$ (unless $G(0,0)=0$), the second and higher derivatives of p with respect to y do not exist at $y=0$. Thus we conclude that

LEMMA 3.1. *Let $p(x, y)$ be the solution of the problem* (3.2). *Then* (a) $\|p\|_{1,\Omega} \leqslant C\,\|g\|_{1,\Omega}$ *for a constant C and* (b) *in general, the partial derivatives*

$$\frac{\partial^n p(x,y)}{\partial y^n} \qquad (n \geqslant 2) \tag{3.8}$$

do not exist on the line $y=0$.

Proof. (a) The inequality follows from (3.5). The statement (b) follows from the above observations. ∎

THEOREM 3.2. *Assume that $[\mathbf{u}, p] \in (H^1(\Omega))^2 \times L^2(\Omega)$ is the solution of* (1.8) *with $\mathbf{u} \equiv 0$, $p \equiv 0$ for $r \geqslant 1$. Then if $\mu > \|A\|\,(\|B\|+1)$, then $\mathbf{u} \in (H^2(\Omega))^2$ and*

$$\|\mathbf{u}\|_{2,\Omega}+\|p\|_{1,\Omega} \leqslant C(\|\mathbf{f}\|_{0,\Omega}+\|g\|_{1,\Omega}) \tag{3.9}$$

for a constant C.

Proof. First letting $G=g-\operatorname{div}\mathbf{u}$ in Lemma 3.1 we have

$$\|p\|_{1,\Omega} \leqslant (\|B\|+1)(\|\mathbf{u}\|_{2,\Omega}+\|g\|_{1,\Omega}). \tag{3.10}$$

Second, letting $F=(\mathbf{f}-\nabla p)/\mu$ in Theorem 3.1 we have

$$\mu\,\|\mathbf{u}\|_{2,\Omega} \leqslant \|A\|\,(\|f\|_{0,\Omega}+\|\nabla p\|_{0,\Omega}). \tag{3.11}$$

Thus, combining (3.10)–(3.11) we get $(\mu-\|A\|\,(\|B\|+1))\,\|\mathbf{u}\|_{2,\Omega} \leqslant C(\|\mathbf{f}\|_{0,\Omega}+\|g\|_{1,\Omega})$. Hence, if $\mu > \|A\|\,(\|B\|+1)$, we have an upper bound for the velocity $\mathbf{u}$ and using it, an upper bound for the pressure p. ∎

Note that Theorem 3.2 was shown for the convex sector (3.1). For a completion of the proof of Theorem 1.5 another case of the convex sector may be considered

$$\Omega=\left\{(x,y) \mid r=\sqrt{x^2+y^2},\ \theta=\tan^{-1}\frac{y}{x},\ 0<r<\infty,\ \omega_1<\theta<\omega_2\right\}, \tag{3.12}$$

where $\pi/2 \leqslant \omega_1 < \pi$, $\pi < \omega_2 \leqslant \frac{3\pi}{2}$. Here $\Gamma_{in} = \varnothing$ and the equation $p_x = G$ in Ω gives

$$p(x, y) = \int_{-\infty}^{x} G(s, y)\, ds. \tag{3.13}$$

Hence, if G is smooth then any partial derivatives of p with respect to the variable y has no jump acrossing the line $y = 0$. Thus, applying the same procedures in the proof of Theorem 3.2 a regularity result like (3.9) can be obtained. Likewise, for any other case of convex sector a result like Theorem 3.2 may be obtained. So Theorem 1.5 is obtained.

4. BOUNDED POLYGONAL DOMAINS

We now consider a bounded polygonal domain D with N vertices. Let P_i $(1 \leqslant i \leqslant N+1)$ be the vertices of D, with $P_{N+1} = P_1$. Let r_i $(1 \leqslant i \leqslant N)$ be the distance from the vertex P_i to other vertices and let $r_0 = \min_{1 \leqslant i \leqslant N} \{r_i\}$. For each $i = 1, ..., N$ let

$$\Omega_i = \left\{ (x, y) \in D : \operatorname{dist}(P_i, (x, y)) < \frac{r_0}{2} \right\} \tag{4.1}$$

and let Ω_0 be be set of points in D which are at distance at least $r_0/4$ from each vertex. Then the collection $\{\Omega_i : 0 \leqslant i \leqslant N\}$ is an open covering of the polygonal domain D. Let χ_i, $i = 0, ..., N$ be a partition of unity subordinate to the covering $\{\Omega_i\}$ (see [1, p. 51]).

Setting $\mathbf{u}_i = \chi_i \mathbf{u}$ and $p_i = \chi_i p$, the solution $[\mathbf{u}, p]$ may be expressed in the form

$$\mathbf{u} = \sum_{i=0}^{N} \mathbf{u}_i, \qquad p = \sum_{i=0}^{N} p_i. \tag{4.2}$$

For each $i = 0, ..., N$ the pair $[\mathbf{u}_i, p_i]$ satisfies the following localized equations

$$\begin{cases} -\mu\, \Delta \mathbf{u}_i + \nabla p_i = \mathbf{f}_i := \chi_i\, \mathbf{f} - 2\,\mu \nabla \chi_i \cdot \nabla \mathbf{u} - \mu \mathbf{u}\, \Delta \chi_i + p\, \nabla \chi_i & \text{in } \Omega_i, \\ \operatorname{div} \mathbf{u}_i + \mathbf{U} \cdot \nabla p_i = g_i := \chi_i g + \mathbf{u} \cdot \nabla \chi_i + p\, \mathbf{U} \cdot \nabla \chi_i & \text{in } \Omega_i, \\ \mathbf{u}_i = 0 \quad \text{on } \Gamma_i, \\ p_i = 0 \quad \text{on } \Gamma_{i, in}, \end{cases} \tag{4.3}$$

where $\Gamma_i := \partial\Omega_i \cap \Gamma$ and $\Gamma_{i,in} := \Gamma_{in} \cap \partial\Omega_i$; if $\Gamma_{i,in} \neq \emptyset$, the inflow boundary condition for pressure will be applied, otherwise it will not. Since $[\mathbf{u}, p] \in (H^1(\Omega))^2 \times L^2(\Omega)$, $[\mathbf{f}_i, g_i] \in (L^2(\Omega_i))^2 \times H^1(\Omega_i)$ for each $i = 0, \ldots, N$. Using (4.3) one may draw conclusions about the regularity of each $[\mathbf{u}_i, p_i]$. Since the support of χ_0 contains a smooth domain, Eq. (4.3) with $i = 0$ may be interpreted as a boundary value problem on this smooth domain, with vanishing boundary conditions. Theorem 1.1 then shows that $[\mathbf{u}_0, p_0] \in (H^2(\Omega))^2 \times H^1(\Omega)$. Similarly, in the case that Ω_i is a convex sector, Theorem 1.5 shows that $[\mathbf{u}_i, p_i] \in (H^2(\Omega))^2 \times H^1(\Omega)$. Finally, if Ω_i is a concave sector, Theorem 1.4 shows that one can write

$$[\mathbf{u}_i, p_i] = [\mathbf{u}_{s,i}, p_{s,i}] + [\mathbf{u}_{r,i}, p_{r,i}]$$

with $[\mathbf{u}_{r,i}, p_{r,i}] \in (H^2(\Omega))^2 \times H^1(\Omega)$ and $\mathbf{u}_{s,i}$ an appropriate singular vector field. Assembling these results with (4.2), we obtain Theorem 1.2. In the case of a convex polygon, Theorem 1.3 is obtained.

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Printed in Belgium